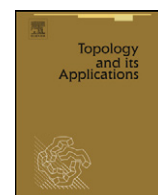


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Topology and its Applications

www.elsevier.com/locate/topolTopological invariants of stable immersions of oriented 3-manifolds in \mathbb{R}^4 C. Casonatto^{a,1}, M.C. Romero Fuster^{b,2}, R. Wik Atique^{c,*,3}^a Faculdade de Matemática, Universidade Federal de Uberlândia, 2121, Uberlândia-MG, CEP 38408-100, Brazil^b Departamento Geometria i Topologia, Facultat de Matemàtiques, Universitat de València, 46100 Burjassot-València, Spain^c Depto. de Matemática, ICMC/USP-São Carlos, Caixa Postal 668, 13560-970, São Carlos, SP, Brazil

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ABSTRACT

We show that the \mathbb{Z} -module of first order local Vassiliev type invariants of stable immersions of oriented 3-manifolds into \mathbb{R}^4 is generated by 3 topological invariants: The number of pairs of quadruple points and the positive and negative linking invariants ℓ^+ and ℓ^- introduced by V. Goryunov (1997) [7]. We obtain the expression for the Euler characteristic of the immersed 3-manifold in terms of these invariants. We also prove that the total number of connected components of the triple points curve is a non-local Vassiliev type invariant.

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1. Introduction

The local topological classification of 3-manifolds stably immersed in \mathbb{R}^4 is given in terms of the branches in general position at the given point. A global classification (up to diffeomorphism) of the stable immersions of a closed 3-manifold M into \mathbb{R}^4 relies first on the introduction of invariants of regular homotopy distinguishing between different connected components of the space $\mathcal{I}mm(M, \mathbb{R}^4)$, in a similar way that the Whitney index distinguishes among different classes of immersed closed plane curves. Such invariants, in the case of the 3-sphere, were introduced by S. Smale (see [13,18]) and admit generalizations to other 3-manifolds [12]. Once we choose a homotopy class, we can consider the concept of stable isotopy in order to classify the stable immersions included in such a class, as done for plane curves by V. Arnol'd [2]. A useful tool for this purpose is the study of Vassiliev type invariants. This technique, pushed forward by V. Vassiliev in [19] in order to obtain topological invariants for knots in \mathbb{R}^3 , follows a strategy that goes back to Poincaré and can be applied to spaces with a stability criterion for which the non-stable objects form a stratified subset of codimension one (called discriminant). The analysis of the incidence relations between the different strata leads to the definition of stable isotopy invariants on the stable objects. Such invariants are calculated in terms of indices and coorientations conveniently attached to the different strata of the discriminant. They can be of different orders, according to the codimension of the strata whose indices affect the invariant.

This method has been successfully applied to several cases of stable maps [2,7,16,17,20,22]. Here we apply it to the case of stable immersions of oriented 3-manifolds into \mathbb{R}^4 . This can be seen as a particular case of the stable maps from 3-manifolds in \mathbb{R}^4 and can be considered as a first step towards its study. The Vassiliev type invariants of a wide class of stable immersions of closed manifolds in Euclidean space have been investigated by T. Ekholm [8,9]. In particular,

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he studied in [8] the first type invariants of 3-knots in \mathbb{R}^5 , obtaining as a by-product some first order invariants for immersions of S^3 into \mathbb{R}^4 . Nevertheless, Ekholm's results do not include a complete determination of the first order invariants for stable immersions of hypersurfaces. On the other hand, V. Goryunov considered in [7] the stable immersions of closed surfaces in 3-space, as a particular case of stable maps, obtaining a complete set of generators for their first class local integer invariants: The variation in their number of pairs of triple points of the image T and in the linking invariant ℓ^- (orientable case). In fact, he extended the definition of the linking invariants, ℓ^- and ℓ^+ , for stable immersions of hypersurfaces in higher dimensions, proving that $\Delta\ell^+$ always vanishes in odd dimensional ambient space. Our contribution in this work consists in determining the dimension of the \mathbb{Z} -module of first order local Vassiliev type invariants for the case of orientable 3-manifolds stably immersed into \mathbb{R}^4 . We obtain a complete set of generators composed of 3 invariants: \mathcal{Q} (number of pairs of quadruple points of the image), ℓ^- and ℓ^+ . We also describe the Euler characteristic of the image as a first order local invariant, given by a convenient combination of the generators. Finally, by refining the stratification of the discriminant in the space of immersions from a 3-manifold to \mathbb{R}^4 , we obtain a new first order non-local invariant given by the number of connected components of the triple points curve.

2. Immersions from 3-manifolds to \mathbb{R}^4 and Vassiliev type invariants

Locally, the image of a stable immersion of a 3-manifold M into \mathbb{R}^4 is either a smooth sheet or the transversal intersection to four smooth sheets.

Non-stable immersions form a *discriminant* hypersurface \mathfrak{D} that separates each one of the connected components of $\mathfrak{Imm}(M, \mathbb{R}^4)$ into different stable isotopy classes of immersions, where two stable immersions are said to be stably isotopic if they can be joined by a path of stable immersions. An integer *stable isotopy invariant* is a locally constant function $I: \mathfrak{Imm}(M, \mathbb{R}^4) - \mathfrak{D} \rightarrow \mathbb{Z}$. This means that for any pair, f and g , of stably isotopic immersions, $I(f) = I(g)$. The discriminant is a stratified subset whose strata Σ^i are made of immersions that appear in generic i -parameter families but can be avoided in generic $(i-1)$ -parameter families. An invariant is said to be *local* if its variation when crossing a given stratum of the discriminant is completely determined by the diffeomorphism type of the local bifurcation of the image at this crossing. An invariant I is said to be of *first order* if given any immersions $f_0, f_1 \in \Sigma^1$ which can be joined by a generic path in $\Sigma^1 \cup \Sigma^2$, we have that $I(f_0) = I(f_1)$.

Vassiliev's technique for defining first order invariants of stable maps works as follows:

1. Attach a coorientation to each codimension one stratum. That is, give a criterium to decide when a transversal path crosses the stratum in the positive or negative direction.
2. Attach a transition index, whose jump is 1 whenever it crosses the stratum S in a positive sense of the coorientation and 0 whenever it crosses any other stratum, to each codimension 1 stratum.
3. A linear combination of indices with coefficients in \mathbb{Z} determines a *Vassiliev order one cocycle* if it satisfies the following *Compatibility condition*: Given a generic closed path (i.e., transversal to all the strata of \mathfrak{D}) around any codimension two stratum S , the sum of the indices corresponding to its transitions through all the codimension one cooriented strata incident to S is 0.

Once we have attached some value of an invariant I to a distinguished stable map f_0 lying in $C^\infty(X, Y)$, the value of I on any other stable map in the same path component than f_0 in $C^\infty(X, Y)$ can be obtained in terms of the indices of the codimension one strata crossed by any generic path γ joining f with f_0 . The compatibility conditions ensure that the value $I(f) \in \mathbb{Z}$ is independent of the chosen generic path.

In order to apply this technique to the stable immersions of a closed 3-manifold into \mathbb{R}^4 , we thus need to describe the codimension one and two phenomena, assign coorientations to those of codimension one and analyze the bifurcation set for those of codimension two. In what follows we shall understand contact in the sense of Montaldi's work [14]. So we associate a contact function to each pair of tangent submanifolds (3-manifolds, or surfaces) whose singularities describe the contact type. Then we say that a contact between two submanifolds is nondegenerate (Morse type) or degenerate, according to whether the corresponding contact function is of Morse type or not. By a tangency between two submanifolds we shall understand that the corresponding contact function is singular at the considered point. The tangency will be degenerate or not, according to this singularity is degenerate or not. The type of tangency means the singularity type.

The multi-germs of immersions with \mathcal{A}_e -codimension 1 have been described by T. Cooper in his PhD thesis [4], see also [5,8]. They are given by

1. Two branches with a nondegenerate tangency at the origin. This tangency may be of elliptic, or hyperbolic type respectively denoted by \mathcal{E} and \mathcal{H} .
2. Three branches with a nondegenerate tangency at the origin of one of them with the surface of intersection of the other two. Again, this tangency may be of elliptic, or hyperbolic type respectively denoted by \mathcal{T}_e and \mathcal{T}_h .
3. Four branches meeting at the origin (pairwise transversal) with a nondegenerate tangency between one of them with the line of intersection of the other three. We denote it by \mathcal{Q} .
4. Five branches meeting transversally at the origin. We denote it by \mathcal{P} .

Their normal forms are the following:

$$\begin{aligned} \mathcal{E}, \mathcal{H} : \begin{cases} (x, y, z, 0) \\ (x, y, z, x^2 + y^2 \pm z^2) \end{cases} & \quad \mathcal{T}_e, \mathcal{T}_h : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, y, z, z + x^2 \pm y^2) \end{cases} \\ \mathcal{Q} : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (x, y, z, y + z + x^2) \end{cases} & \quad \mathcal{P} : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (0, x, y, z) \\ (x, y, z, x + y + z). \end{cases} \end{aligned}$$

In what follows we describe the codimension two strata.

Theorem 2.1. *The following list describes all the \mathcal{A} -classes of immersed multi-germs of \mathcal{A}_e -codimension 2 from \mathbb{R}^3 to \mathbb{R}^4 .*

1. Two branches with a type A_2 degenerate tangency at the origin. This tangency may be of elliptic, or hyperbolic type respectively denoted as B_2^+ and B_2^- .
2. Three branches with a type A_2 degenerate tangency at the origin of one of them with the surface of intersection of the other two. We denote it by T^d .
3. Three branches with a nondegenerate tangency at the origin of two of them. We denote it by $T_2^{\pm\pm}$.
4. Four branches meeting at the origin (pairwise transversal) with a type A_2 degenerate tangency between one of them with the line of intersection of the other three. We denote it by Q^d .
5. Four branches meeting at the origin with a nondegenerate tangency of one of them with the surface of intersection of other two. We denote it by Q_2^{\pm} .
6. Five branches meeting at the origin (pairwise transversal) with a nondegenerate tangency between one of them with the line of intersection of other three. We denote it by P_2 .

The corresponding normal forms are the following:

$$\begin{aligned} B_2^{\pm} : \begin{cases} (x, y, z, 0) \\ (x, y, z, x^2 \pm y^2 + z^3) \end{cases} & \quad T^d : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, y, z, z + x^2 + y^3) \end{cases} & \quad T_2^{\pm\pm} : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, y, z, \pm x^2 + y^2 \pm z^2) \end{cases} \\ Q^d : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (x, y, z, y + z + x^3) \end{cases} & \quad Q_2^{\pm} : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (x, y, z, z \pm x^2 + y^2) \end{cases} & \quad P_2 : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (0, x, y, z) \\ (x, y, z, x^2 + y + z) \end{cases} \end{aligned}$$

The germ T_2^{--} is equivalent to T_2^{++} .

Proof. Using the same argument as Mond [15], bi-germs of immersions have the form $f : \begin{cases} (x, y, z, 0) \\ (x, y, z, \varphi(x, y, z)) \end{cases}$.

The \mathcal{A} -class of f is given by the \mathcal{K} -class of φ , where \mathcal{K} is the contact group. Moreover, the \mathcal{A}_e -codimension of f is equal to the \mathcal{K}_e -codimension of φ . So we have B_2^{\pm} .

Similarly to [21, Proposition 3.4], a tri-germ of immersions takes the form

$$f : \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, y, z, p(x, y, z)) \end{cases}$$

and the \mathcal{A} -class of f is given by the ${}_V\mathcal{K}$ -class of p , where ${}_V\mathcal{K}$ is a subgroup of the contact group that preserves the plane $V = \{z = 0\}$ in the source, following notation of J. Damon [6]. Moreover, the \mathcal{A}_e -codimension of f is equal to the ${}_V\mathcal{K}_e$ -codimension of p . In this case the ${}_V\mathcal{K}$ classification was obtained by Arnol'd [1] (known as boundary singularity). So we have T^d and $T_2^{\pm\pm}$.

In order to obtain the remaining normal forms we use [21, Theorem 2.1] that says that the \mathcal{A} -class of a multi-germ $f : (\mathbb{R}^n, S) \rightarrow (\mathbb{R}^{n+1}, 0)$ of corank 1 is given by the ${}_V\mathcal{K}$ -class of the function-germ that defines the image of one of the branches of f and V is the image of the remaining branches.

Lets first consider V as the union of three hyperplanes in \mathbb{R}^4 meeting at the origin and two-by-two transversal. Denoting (X, Y, Z, W) as coordinates in the target, V is given by $YZW = 0$. The module of logarithmic vector fields tangent to V , $\text{Derlog } V$, is generated by $\partial/\partial X$, $Y\partial/\partial Y$, $Z\partial/\partial Z$ and $W\partial/\partial W$.

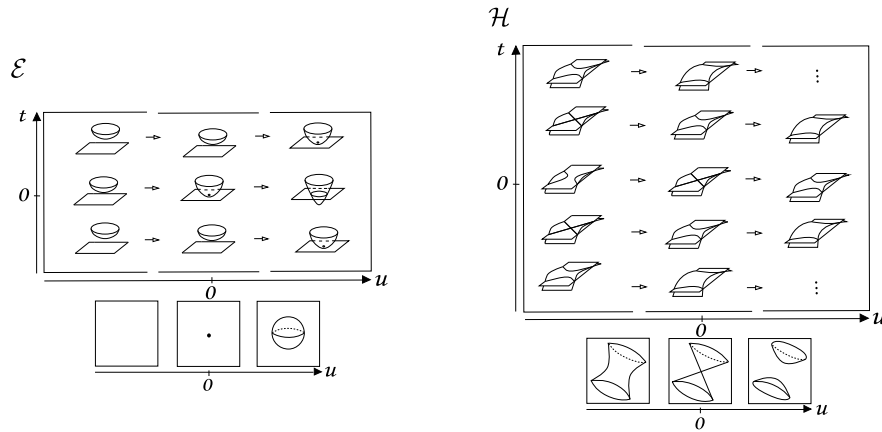


Fig. 1. Transitions through strata \mathcal{E} and \mathcal{H} .

The tangent space to the ${}_V\mathcal{K}$ -orbit of a germ $h: (\mathbb{R}^4, 0) \rightarrow \mathbb{R}$ is given by: $th(\text{Derlog } V \cap \mathcal{M}_4\theta(4)) + \langle h \rangle$.

Now applying the complete transversal method [3] we obtain Q^d and Q_2^\pm .

The same technique is applied when V is the union of four hyperplanes in a general position in \mathbb{R}^4 to complete the proof. \square

Remark. By using the method described above one can find the non-simple 6-germ of \mathcal{A}_e -codimension 5:

$$\mathcal{S}: \begin{cases} (x, y, z, 0) \\ (x, y, 0, z) \\ (x, 0, y, z) \\ (0, x, y, z) \\ (x, y, z, x + y + z) \\ (x, y, z, ax + by + cz) \end{cases}$$

where a, b, c are modalities and $a - 1, b - 1, c - 1, a - b, a - c, b - c$ are non-zero. Considered as a union of orbits, obtained when a, b and c vary, \mathcal{S} is a codimension two stratum (i.e. in Σ^2).

3. Analysis of codimension one strata

3.1. Coorientation of codimension one strata

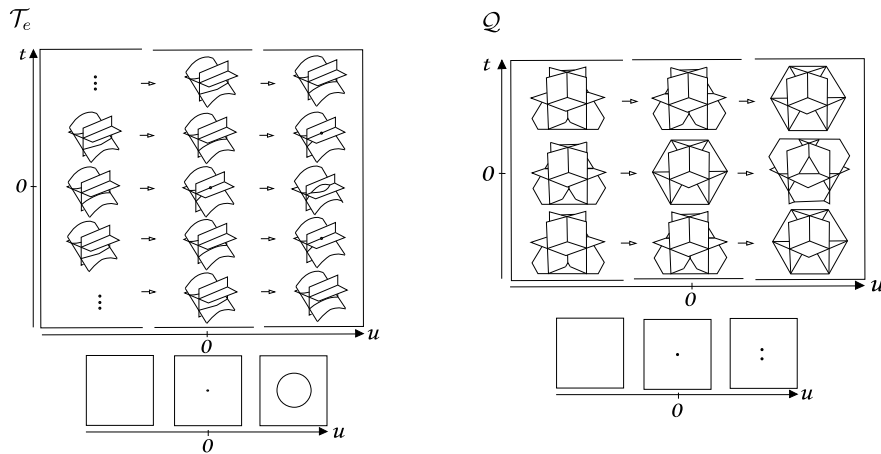
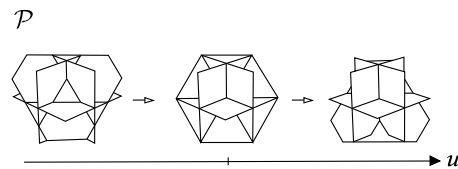
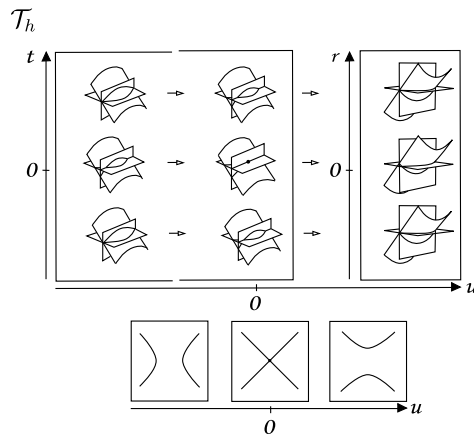
In order to establish the coorientation of each codimension one stratum, we shall use some criteria relative to the variation in the image of the immersion when traverses such stratum.

Figs. 1 and 2 illustrate the effect of the transitions through the different codimension one strata. In all the cases, the upper figure represents the transitions of different 3-dimensional sections depending on an internal parameter t . The variable u corresponds to the parameter of the considered transversal path (1-parameter family). The lower drawings in Fig. 1 represent the effect of each transition in the local topology of the set of double points of the immersion. The lower drawings in Fig. 2 represent, respectively, the variation in the curve of triple points and in the quadruple points.

We fix the coorientation on these strata as follows:

1. The positive direction at \mathcal{E} and \mathcal{H} is determined in the sense that the number of connected components in the surface of double points increases.
2. The positive direction at \mathcal{T}_e is determined in the sense that the number of connected components of the curve of triple points increases.
3. The positive direction at \mathcal{Q} is determined in the sense that the number of quadruple points increases.

In order to visualize the local effects of the transition \mathcal{P} we shall use a different technique: We linearize one of the branches and consider the section determined by the other branches in this one. In this case we obtain a picture of four hypersurfaces intersecting in \mathbb{R}^3 , in which triple and double points respectively represent quadruple and triple points of the original object. The analysis of the behaviour of the former in this section provides information enough on the behaviour of the later in \mathbb{R}^4 . Taking these observations into account, we can represent the transition through the stratum \mathcal{P} as in Fig. 3, which corresponds to Goryunov's transition \mathcal{Q} in [7]. Here, we observe that the two 3-simplices obtained before and after the crossing in the 3-dimensional picture, actually represent 4-simplices in the 4-dimensional picture.

Fig. 2. Transitions through strata \mathcal{T}_e and \mathcal{Q} .Fig. 3. Transitions through stratum \mathcal{P} .Fig. 4. Transitions through stratum \mathcal{T}_h .

We can deduce from the above Figs. 3 and 4 that there are no topological differences from one side to the other of the transitions through the strata \mathcal{T}_h and \mathcal{P} . Therefore they are not coorientable. Nevertheless, we shall see below how to sub-stratify the stratum \mathcal{P} in such a way that the different components become coorientable.

3.2. Sub-stratification of codimension one strata

If the 3-manifold M is oriented, the orientation induced on the different branches at multiple points induces a sub-stratification of some of the codimension one strata. The criterion used here is analogous to the one followed by V. Goryunov in [7] for the case of surfaces in \mathbb{R}^3 . Given a multi-germ $f : (\mathbb{R}^3, S) \rightarrow (\mathbb{R}^4, 0)$, for each branch f_i of f we consider the normal vector in the direction induced by the orientation of M at the considered point (in such a way that a positive basis of the tangent space, followed by this vector defines a positive basis for \mathbb{R}^4). We observe that for each one of the strata \mathcal{E} , \mathcal{T}_e , \mathcal{Q} and \mathcal{P} appears a topological 3-sphere in the right-hand side of the transition (in the positive direction of the coorientation) which is composed by different pieces of hypersurfaces lying in the different branches. We shall distinguish among the different possibilities arising from the different combinations of orientation on each branch. In order to visualize this better, we can cut the 3-sphere by a transversal slice (hyperplane) and obtain a 3-dimensional section containing a topological 2-sphere composed by pieces of surfaces lying in the sections of the different branches. Each one of this

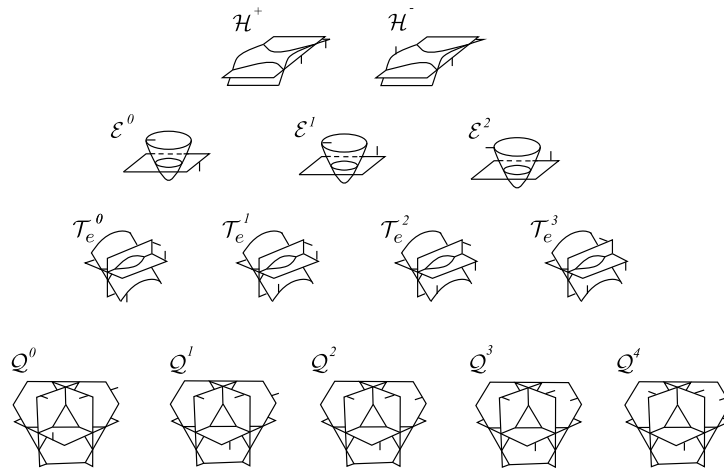


Fig. 5. Image of the 3-manifolds in the section corresponding to $t = 0$ and parameter $u > 0$.

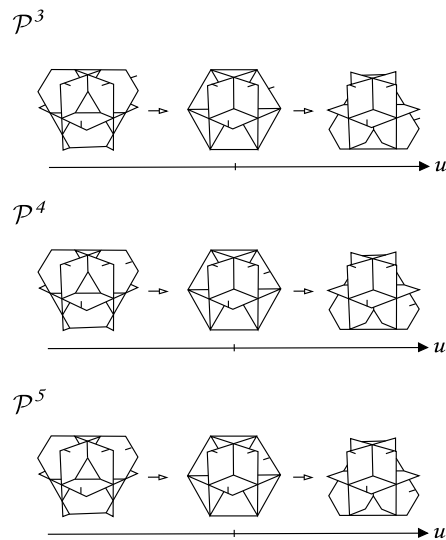


Fig. 6. Coorientation of the strata \mathcal{P}^3 , \mathcal{P}^4 and \mathcal{P}^5 .

sections contains the information on the orientation on the whole branch, so it is enough to distinguish among these 3-dimensional cases. Following Goryunov we say that a piece is *positive* if its positive normal points towards the exterior of the 2-sphere. Otherwise, we call it *negative piece*. According to this we obtain the following substrata: \mathcal{E}^j , $j = 0, 1, 2$, \mathcal{T}_e^j , $j = 0, 1, 2, 3$, \mathcal{Q}^j , $j = 0, 1, 2, 3, 4$ and \mathcal{P}^j , $j = 3, 4, 5$, where j denotes the number of positive pieces composing the corresponding 2-sphere.

In the case of \mathcal{H} we analyze the possible coorientations at the image of the section corresponding to $t = 0$ and we obtain two possibilities, \mathcal{H}^+ and \mathcal{H}^- : the branches, at the tangency point, have coinciding, respectively opposite, coorientations (see Fig. 5).

We have now a criterion to coorient the stratum \mathcal{P} : The normals to the 5 higher dimensional proper faces of the 4-simplex formed at each side of the stratum \mathcal{P}^j , $j = 3, 4, 5$, may point inwards or outwards according to the orientation of each of one of the branches involved. Then the number of positive and negative faces may vary when crossing each stratum. We shall coorient each stratum \mathcal{P}^j , $j = 3, 4, 5$, towards the side at which the number of positives faces of the 4-simplex increases (see Fig. 6).

Similar arguments to those used by Goryunov in [7] for sub-stratifying the stratum \mathcal{H} could be applied to sub-stratify our stratum \mathcal{T}_h . However this sub-stratification does not seem to be natural in our case. Moreover, after calculating bifurcation diagrams, it leads to certain compatibility conditions implying that the transition index at each \mathcal{T}_h sub-stratum is zero. So for the sake of simplicity, here \mathcal{T}_h is non-coorientable.

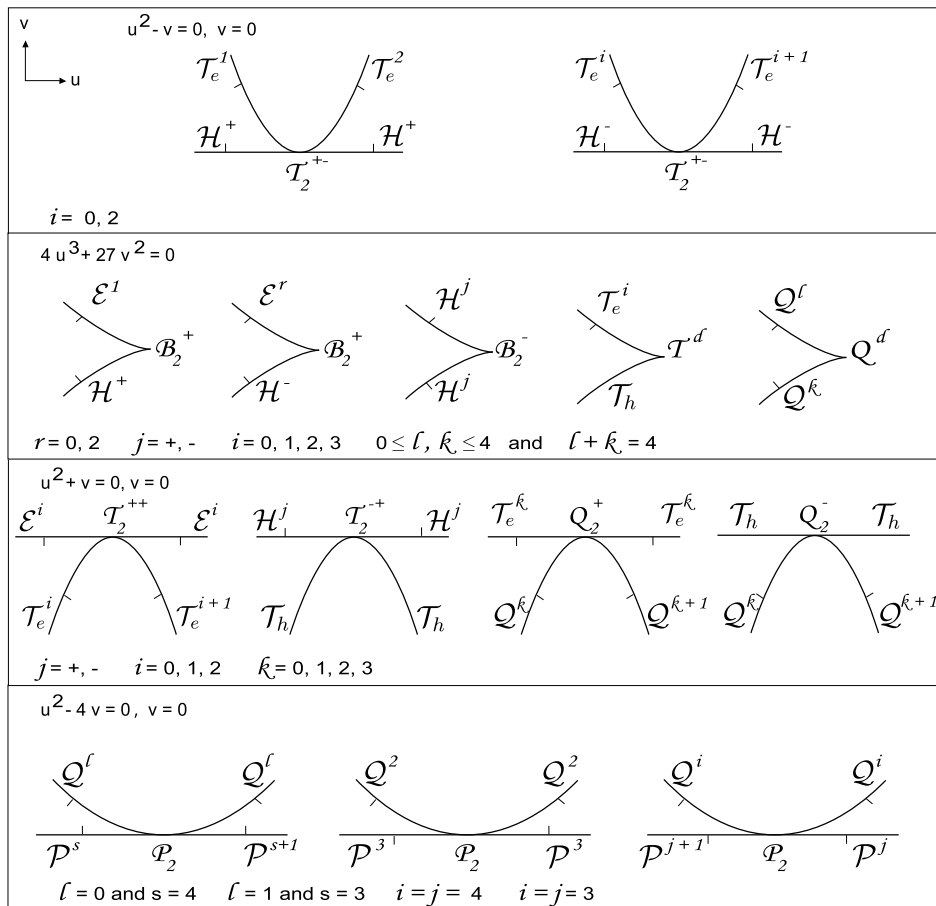


Fig. 7. Bifurcation diagrams of the codimension two strata.

4. Bifurcation diagrams of codimension two strata

Let $f : X \rightarrow Y$ be a continuous map of topological spaces. The k -fold multiple point space of f ($1 \leq k < \infty$) is:

$$D^k(f) = \text{closure}\{(x_1, \dots, x_k) \in X^k / f(x_1) = \dots = f(x_k), x_i \neq x_j \text{ if } i \neq j\}.$$

We say that a multi-germ $f : (\mathbb{R}^n, S) \rightarrow (\mathbb{R}^{n+1}, 0)$ is of \mathcal{K} -type $A_{(k_1, \dots, k_s)}$, $s = |S|$, if f has an A_{k_i} -singularity at the i -th source point. An immersed multi-germ is of type $A_{(0, \dots, 0)}$. We denote by $G_{(k_1, \dots, k_s)}$ the map whose 0-set is the closure of $A_{(k_1, \dots, k_s)}$. The s -fold multiple point space $D^s(f)$ is the closure of the set $A_{(0, \dots, 0)}$. By a result of Mather, the stable corank 1 multi-germs are those transverse to their \mathcal{K} -class $A_{(k_1, \dots, k_s)}$, that is, the maps $G_{(k_1, \dots, k_s)}$ are submersions.

If $F(x, u) = (f_u(x), u)$ is a versal unfolding of a multi-germ f , the bifurcation diagram of f is the set of parameters u such that f_u is not stable. Therefore, for an immersed multi-germ $f : (\mathbb{R}^3, S) \rightarrow (\mathbb{R}^4, 0)$ the bifurcation diagram consists of the parameters u for which the maps defining $D^k(f_u)$ are not submersive, for all k .

We calculate the bifurcation diagram of B_2^\pm (see Fig. 7). The remaining is similar. The deformation is given by

$$f_{u,v} : \begin{cases} (x, y, z, 0) \\ (x, y, z, x^2 \pm y^2 + z^3 + uz + v). \end{cases}$$

The 2-fold multiple point space of $f_{u,v}$ is the only one non-empty. It is the closure of

$$\{(x, y, z, \bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^6 / x = \bar{x}, y = \bar{y}, z = \bar{z}, x^2 \pm y^2 + z^3 + uz + v = 0\},$$

so $G_{(0,0)} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by $G(x, y, z) = x^2 \pm y^2 + z^3 + uz + v$, and is not submersive when $x = y = 0$ and $3z^2 + u = 0$. Therefore the conditions are:

$$x = y = 3z^2 + u = x^2 \pm y^2 + z^3 + uz + v = 0,$$

and then the bifurcation diagram is $\mathcal{B}(B_2^\pm) = \{(u, v) \in \mathbb{R}^2 / 4u^3 + 27v^2 = 0\}$.

The bifurcation diagram of the 6-germ \mathcal{S} consists of 6 lines passing through the origin of \mathbb{R}^2 . The only codimension one stratum that incides in \mathcal{S} is \mathcal{P} .

Remark. Notice that the coorientation of the stratum \mathcal{E}^0 on the B_2^+ bifurcation diagram involving \mathcal{E}^r and \mathcal{H}^- coincides with that of \mathcal{E}^2 , and in the opposite direction to the one pointed in Goryunov's paper [7]. This is due to the fact that in our case the coorientation of \mathcal{H}^- is given, regardless the orientation of the 3-manifold, by the topology of double points set (see Fig. 1). Whereas in Goryunov's case in order to coorient this stratum one needs to consider the orientation of the domain. As a consequence, we get that $\eta_{\mathcal{E}^0} = \eta_{\mathcal{E}^2}$ for immersions of 3-manifolds into \mathbb{R}^4 (see Section 5), differently from that of surfaces into \mathbb{R}^3 where $\eta_{\mathcal{E}^0} = -\eta_{\mathcal{E}^2}$.

5. The module of first order local Vassiliev type invariants

5.1. Compatibility conditions and determination of generators

In this section we calculate the compatibility conditions by using the information about bifurcation diagrams obtained in Section 4.

Any integer first order local invariant may be seen as a linear combination of indices attached to the different codimension one strata. In fact, we have that the increment ΔI of an invariant I along a generic path γ in $\mathcal{Imm}(M, \mathbb{R}^4)$ is given by $\Delta I = \sum_i \eta_{S_i} \Delta S_i$, where $\eta_{S_i} \in \mathbb{Z}$, S_i are the codimension one strata and ΔS_i denotes the index of S_i . So we can write,

$$\begin{aligned} \Delta I = & \eta_{\mathcal{E}^0} \Delta \mathcal{E}^0 + \eta_{\mathcal{E}^1} \Delta \mathcal{E}^1 + \eta_{\mathcal{E}^2} \Delta \mathcal{E}^2 + \eta_{\mathcal{H}^+} \Delta \mathcal{H}^+ + \eta_{\mathcal{H}^-} \Delta \mathcal{H}^- + \eta_{\mathcal{T}_e^0} \Delta \mathcal{T}_e^0 \\ & + \eta_{\mathcal{T}_e^1} \Delta \mathcal{T}_e^1 + \eta_{\mathcal{T}_e^2} \Delta \mathcal{T}_e^2 + \eta_{\mathcal{T}_e^3} \Delta \mathcal{T}_e^3 + \eta_{\mathcal{T}_h} \Delta \mathcal{T}_h + \eta_{\mathcal{Q}^0} \Delta \mathcal{Q}^0 + \eta_{\mathcal{Q}^1} \Delta \mathcal{Q}^1 \\ & + \eta_{\mathcal{Q}^2} \Delta \mathcal{Q}^2 + \eta_{\mathcal{Q}^3} \Delta \mathcal{Q}^3 + \eta_{\mathcal{Q}^4} \Delta \mathcal{Q}^4 + \eta_{\mathcal{P}^3} \Delta \mathcal{P}^3 + \eta_{\mathcal{P}^4} \Delta \mathcal{P}^4 + \eta_{\mathcal{P}^5} \Delta \mathcal{P}^5. \end{aligned} \quad (1)$$

From the compatibility conditions obtained by analyzing all the bifurcation diagrams, we obtain the following relations between the different η_{S_i} :

1. $\eta_{\mathcal{E}^0} = \eta_{\mathcal{E}^2} = \eta_{\mathcal{H}^-}$;
2. $\eta_{\mathcal{E}^1} = \eta_{\mathcal{H}^+}$;
3. $\eta_{\mathcal{T}_e^0} = \eta_{\mathcal{T}_e^1} = \eta_{\mathcal{T}_e^2} = \eta_{\mathcal{T}_e^3} = \eta_{\mathcal{T}_h}$;
4. $\eta_{\mathcal{Q}^0} = \eta_{\mathcal{Q}^1} = \eta_{\mathcal{Q}^2} = \eta_{\mathcal{Q}^3} = \eta_{\mathcal{Q}^4}$;
5. $\eta_{\mathcal{P}^3} = \eta_{\mathcal{P}^4} = \eta_{\mathcal{P}^5} = 0$;
6. $\eta_{\mathcal{T}_h} = 0$.

By substituting these relations in Eq. (1) we obtain the following result.

Theorem 5.1. Any one-cocycle, i.e. any integer-valued function I satisfying $\Delta I = 0$ on every generic contractible closed path in $\mathcal{Imm}(M, \mathbb{R}^4)$, where M is an oriented closed 3-manifold, can be written, up to an additive constant, as a linear combination of the following three generators:

1. $\Delta I_1 = \Delta \mathcal{Q}$;
2. $\Delta I_2 = \Delta \mathcal{E}^0 + \Delta \mathcal{E}^2 + \Delta \mathcal{H}^-$;
3. $\Delta I_3 = \Delta \mathcal{E}^1 + \Delta \mathcal{H}^+$.

We observe that ΔI_1 is the increment in the number of pairs of quadruple points of the immersion. Moreover, ΔI_2 and ΔI_3 , which deal respectively with double inverse and direct self-tangencies of the image, coincide with the linking invariants ℓ^+ and ℓ^- described by Goryunov in [7]. So we can state:

Corollary 5.1.1. A complete set of generators for space of integer local invariants of smooth mappings of an oriented closed 3-manifold into \mathbb{R}^4 is the following: $I_{\mathcal{Q}}$ (number of pairs of quadruple points), I_{ℓ^-} , I_{ℓ^+} .

We see next how to obtain the Euler characteristic of the image of a stable immersion in terms of first order local invariants.

Theorem 5.2. The Euler characteristic of the image of a stable immersion of an oriented closed 3-manifold into \mathbb{R}^4 is a first order local Vassiliev type invariant, whose expression in terms of the above generators is given by $\Delta I_{\chi} = -2(\Delta I_1 + \Delta I_2 + \Delta I_3)$.

Proof. We must check that the variation of the Euler characteristic of the image when crossing each codimension one stratum coincides with that of I_{χ} . In order to study the local variation of the Euler characteristic we use the method

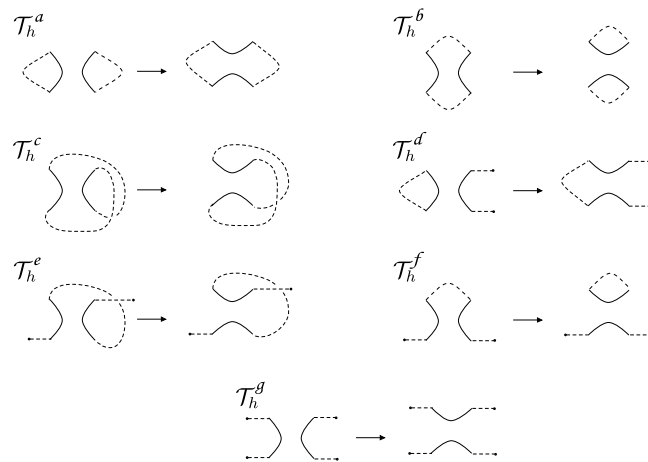


Fig. 8. Sub-stratification of \mathcal{T}_h strata.

described by Cooper, Mond and Wik-Atique in [5]: Let $f : (\mathbb{R}^n, S) \rightarrow (\mathbb{R}^{n+1}, 0)$ be a multi-germ of \mathcal{A}_e -codimension 1 and f_u its stable deformation. Up to homeomorphism, there are two (possibly equivalent) choices for the image of f_u : one for positive u and one for negative u . We shall call these $D^+(f)$ and $D^-(f)$, respectively. Let $A_f(u, x) = (u, f_{u^2}(x))$ be the augmentation of f , then A_f has \mathcal{A}_e -codimension 1 and

- (i) $D^+(A_f)$ is homotopy equivalent to $D^+(f)$,
- (ii) $D^-(A_f)$ is homotopy equivalent to the suspension of $D^-(f)$, denoted by $S(D^-(f))$.

Considering f as one of E, H, T or Q (as in Goryunov's work [7]), then A_f is either $\mathcal{E}, \mathcal{H}, \mathcal{T}_e$ or \mathcal{Q} respectively. We observe that the only codimension one transitions affecting the Euler characteristic of the local image of the immersion are $\mathcal{E}, \mathcal{H}, \mathcal{T}_e$ and \mathcal{Q} , because the positive and negative sides of the transitions of \mathcal{P} and \mathcal{T}_h are locally diffeomorphic.

Since $D^-(E) = D^-(T) = D^-(Q) = D^+(Q) = S^2$, $D^-(H) = D^+(H) = D^+(T) = S^1$ and $D^+(E) = \{p_1\} \cup \{p_2\}$, where S^i is a topological i -sphere and p_1, p_2 are points, we have

- $\Delta\mathcal{X}(\mathcal{E}) = \mathcal{X}(D^-(\mathcal{E})) - \mathcal{X}(D^+(\mathcal{E})) = \mathcal{X}(S(D^-(E))) - \mathcal{X}(D^+(E)) = \mathcal{X}(S^3) - \mathcal{X}(\{p_1\} \cup \{p_2\}) = -2$;
- $\Delta\mathcal{X}(\mathcal{H}) = \mathcal{X}(D^+(H)) - \mathcal{X}(S(D^-(H))) = \mathcal{X}(S^1) - \mathcal{X}(S^2) = -2$;
- $\Delta\mathcal{X}(\mathcal{T}_e) = \mathcal{X}(S(D^-(T))) - \mathcal{X}(D^+(T)) = \mathcal{X}(S^3) - \mathcal{X}(S^1) = 0$;
- $\Delta\mathcal{X}(\mathcal{Q}) = \mathcal{X}(S(D^-(Q))) - \mathcal{X}(D^+(Q)) = \mathcal{X}(S^3) - \mathcal{X}(S^2) = -2$.

Therefore all these variations of the Euler characteristic of the branch set coincide with those of I_χ . \square

According to [11], we have $\chi(f(M)) = -\frac{1}{2}\chi(D^2(f))$. So we get from the above theorem that $\chi(D^2(f))$ is also a first order local Vassiliev type invariant, given by $\Delta I_{\chi(D^2(f))} = 4(\Delta I_1 + \Delta I_2 + \Delta I_3)$.

Remark. The invariant I_Q was also obtained by T. Ekholm in [8]. He also determined other two topological invariants given in terms of resolutions of double and triple points sets of the immersions. As we shall see in the next section, the invariants related with the triple points are of non-local type.

Finally, we observe that the above method can be applied to study the Vassiliev type invariants of stable fold maps of 4-manifolds into \mathbb{R}^4 . In this case the image of the singular set behaves as a stably immersed 3-manifold. Then the above considerations can be easily translated in this context to deduce that the \mathbb{Z} -module of first order local invariants of such maps is 3-dimensional, a system of generators being given by the quadruple points and the two linking invariants on the discriminant set. Notice that the space of stable fold maps of 4-manifolds into is not path connected and the Vassiliev type invariants are defined modulo constants on each path component.

6. A non-local Vassiliev type invariant

The number of connected components of the curve of triple points is a topological invariant for stable immersions of 3-manifolds into \mathbb{R}^4 . But we observe that it is not of local type, for its variation when crossing the codimension one stratum \mathcal{T}_h does not provide a complete information on its global behaviour. Nevertheless, we can refine the stratification of the space of codimension one immersions in a way to obtain this number as a first order Vassiliev type invariant. To do this

we shall proceed analogously as K. Houston [10] did to obtain an extended system of invariants for stable maps of closed orientable surfaces into \mathbb{R}^3 including the number of connected components of the curve of double points.

By taking into account that the curve of triple points may either form loops or end at quadruple points, we can sub-stratify \mathcal{T}_h into seven sub-strata (see Fig. 8). However, just four of them are coorientable, namely $\mathcal{T}_h^a, \mathcal{T}_h^b, \mathcal{T}_h^d$ and \mathcal{T}_h^f (the positive direction is determined in the sense that the number of connected components of the curve of triple points increases). In Fig. 8 the continuous lines represent the local changes of the curve of triple points and dotted lines describe its possible global behaviour.

Observe that crossing through \mathcal{T}_h^a or \mathcal{T}_h^d is respectively opposite to crossing through \mathcal{T}_h^b or \mathcal{T}_h^f .

By applying again Vassiliev's technique to this sub-stratification of \mathcal{T}_h , we obtain a new generator

$$\Delta I_4 = \Delta \mathcal{T} + \Delta \mathcal{T}_h^b + \Delta \mathcal{T}_h^f - \Delta \mathcal{T}_h^a - \Delta \mathcal{T}_h^d$$

which coincides with the increment in the number of connected components of the curve of triple points.

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